4.5 Rolle's Theorem

4.5.1 Definition

Let f be a real valued function defined on the closed interval $[a, b]$ such that,

(1) $f(x)$ is continuous in the closed interval $[a, b]$

(2) $f(x)$ is differentiable in the open interval $]a,b[$ and

(3) $f(a) = f(b)$

Then there is atleast one value c of x in open interval $]a, b[$ for which $f'(c) = 0$.

4.5.2 Analytical Interpretation

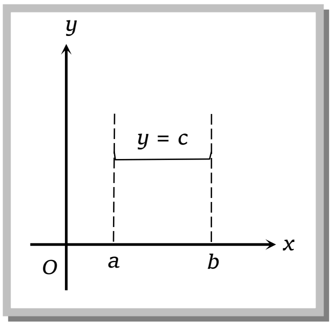
Now, Rolle's theorem is valid for a function such that

(1) $f(x)$ is continuous in the closed interval $[a, b]$

(2) $f(x)$ is differentiable in open interval $]a, b[$ and

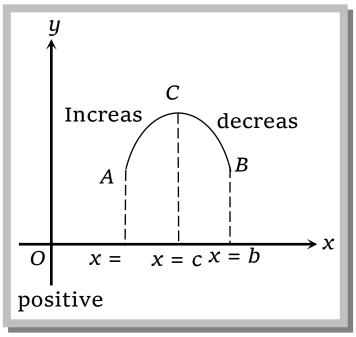
(3) $f(a) = f(b)$

So, generally two cases arises in such circumstances.



Case I: $f(x)$ is constant in the interval $[a, b]$ then $f'(x) = 0$ for all $x \in [a,b]$. Hence, Rolle's theorem follows, and we can say, $f'(c) = 0$, where $a < c < b$

Case II: $f(x)$ is not constant in the interval $[a, b]$ and since $f(a) = f(b)$.



The function should either increase or decrease when x takes values slightly greater than a.

Now, let $f(x)$ increases for $x > a$

Since, $f(a) = f(b)$, hence the function must seize to increase at some value $x = c$ and decreasing upto $x = b$.

Clearly at $x = c$ function has maximum value.

Now let h be a small positive quantity then, from definition of maximum value of the function,

$$f(c+h) - f(c) < 0 \quad \text{and} \quad f(c-h) - f(c) < 0$$

$$\therefore \quad \frac{f(c+h) - f(c)}{h} < 0 \text{ and } \frac{f(c-h) - f(c)}{-h} > 0$$

So, $$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \leq 0 \text{ and } \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} \geq 0 \qquad .....(i)$$

But, if $$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \neq \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h},$$

The Rolle's theorem cannot be applicable because in such case,

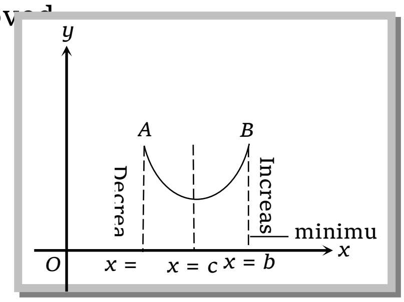
RHD at $x = c \neq \text{LHD}$ at $x = c$.

Hence, $f(x)$ is not differentiable at $x = c$, which contradicts the condition of Rolle's theorem.

$\therefore$ Only one possible solution arises, when $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = 0$

Which implies that, $f'(c) = 0$ where $a < c < b$

Hence, Rolle's theorem is proved.



Similarly, the case where $f(x)$ decreases in the interval $a < x < c$ and then increases in the interval $c < x < b, f'(c) = 0$. But when $x = c$, the minimum value of $f(x)$ exists in the interval [a, b].

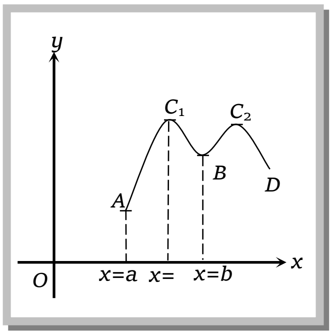
4.5.3 Geometrical Interpretation

Consider the portion AB of the curve $y = f(x)$, lying between $x = a$ and $x = b$, such that

(1) It goes continuously from A to B.

(2) It has tangent at every point between A and B and

(3) Ordinate of A = ordinate of B



From figure, it is clear that $f(x)$ increases in the interval AC_1, which implies that $f'(x) > 0$ in this region and decreases in the interval C_1B which implies $f'(x)<0$ in this region. Now, since there is unique tangent to be drawn on the curve lying in between A and B and since each of them has a unique slope *i.e.,* unique value of $f'(x)$.

$\therefore$ Due to continuity and differentiability of the function $f(x)$ in the region A to B. There is a point $x=c$ where $f'(c)=0$. Hence, $f'(c)=0$ where $a<c<b$

Thus Rolle's theorem is proved.

Similarly the other parts of the figure given above can be explained, establishing Rolle's theorem throughout.

Note : On Rolle's theorem generally two types of problems are formulated.

- ❑ To check the applicability of Rolle's theorem to a given function on a given interval.
- ❑ To verify Rolle's theorem for a given function in a given interval.

In both types of problems we first check whether $f(x)$ satisfies the condition of Rolle's theorem or not.

The following results are very helpful in doing so.

(i) A polynomial function is everywhere continuous and differentiable.

(ii) The exponential function, sine and cosine functions are everywhere continuous and differentiable.

(iii) Logarithmic functions is continuos and differentiable in its domain.

(iv) tan x is not continuous and differentiable at $x=\pm\frac{\pi}{2},\pm\frac{3\pi}{2}, \pm\frac{5\pi}{2},.......$

(v) $|x|$ is not differentiable at $x=0$.

(vi) If $f'(x)$ tends to $\pm\infty$ as $x\to K$, then $f(x)$ is not differentiable at $x=K$.

For example, if $f(x)=(2x-1)^{1/2}$, then $f'(x)=\frac{1}{\sqrt{2x-1}}$ is such that as $x\to\left(\frac{1}{2}\right)^+ \Rightarrow f'(x)\to\infty$

So, $f(x)$ is not differentiable at $x=\frac{1}{2}$.

Example: 1 The function $f(x)=x(x+3)e^{-1/2x}$ satisfies all the condition of Rolle's theorem in [– 3, 0]. The value of c is

(a) 0 (b) 1 (c) – 2 (d) – 3

Solution: (c) To determine 'c' in Rolle's theorem, $f'(c)=0$

Here $f'(x)=(x^2+3x)e^{-(1/2)x}.\left(-\frac{1}{2}\right)+(2x+3)e^{-(1/2)x} = e^{-(1/2)x}\left\{-\frac{1}{2}(x^2+3x)+2x+3\right\} = -\frac{1}{2}e^{-(x/2)}\{x^2-x-6\}$

$\therefore$ $f'(c)=0 \Rightarrow c^2-c-6=0 \Rightarrow c=3,-2$.

But $c=3\notin[-3,0]$, Hence c = –2.

Example: 2 If the function $f(x)=x^3-6x^2+ax+b$ satisfies Rolle's theorem in the interval [1, 3] and $f'\left(\frac{2\sqrt{3}+1}{\sqrt{3}}\right)=0$ then

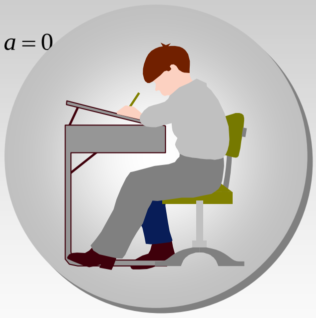
[MP PET 2002]

(a) $a=11$ (b) $a=-6$ (c) $a=6$ (d) $a=1$

Solution: (a) $f(x)=x^3-6x^2+ax+b \Rightarrow f'(x)=3x^2-12x+a$

$$\Rightarrow f'(c)=0 \Rightarrow f'\left(2+\frac{1}{\sqrt{3}}\right)=0 \Rightarrow 3\left(2+\frac{1}{\sqrt{3}}\right)^2-12\left(2+\frac{1}{\sqrt{3}}\right)+a=0$$

$$\Rightarrow 3\left(4+\frac{1}{3}+\frac{4}{\sqrt{3}}\right)-12\left(2+\frac{1}{\sqrt{3}}\right)+a=0 \Rightarrow 12+1+4\sqrt{3}-24-4\sqrt{3}+a=0$$



Assignment

Rolle's Theorem

Basic Level

1. Rolle's theorem is true for the function $f(x)=x^2-4$ in the interval

(a) [– 2, 0] (b) [– 2, 2] (c) $\left[0, \frac{1}{2}\right]$ (d) [0, 2]

2. For which interval, the function $\frac{x^2-3x}{x-1}$ satisfies all the conditions of Rolle's theorem

(a) [0, 3] (b) [–3, 0] (c) [1.5, 3] (d) For no interval

3. If $f(x)$ satisfies the conditions of Rolle's theorem in [1, 2] and $f(x)$ is continuous in [1, 2] then $\int_1^2 f'(x)dx$ is equal to **[DCE 2002]**

(a) 3 (b) 0 (c) 1 (d) 2

4. Consider the function $f(x)=e^{-2x}\sin 2x$ over the interval $\left(0, \frac{\pi}{2}\right)$. A real number $c \in \left(0, \frac{\pi}{2}\right)$, as guaranteed by Roll's theorem, such that $f'(c)=0$ is

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

5. If the function $f(x)=ax^3+bx^2+11x-6$ satisfies the conditions of Rolle's theorem for the interval [1, 3] and $f'\left(2+\frac{1}{\sqrt{3}}\right)=0$, then the values of a and b are respectively

(a) 1, – 6 (b) – 2, 1 (c) –1, $\frac{1}{2}$ (d) – 1, 6

6. Rolle's theorem is not applicable to the function $f(x) = |x|$ defined on [– 1, 1] because **[AISSE 1986; MP PET 1994, 95]**

(a) f is not continuous on [– 1, 1]
(b) f is not differentiable on (– 1, 1)
(c) $f(-1) \neq f(1)$
(d) $f(-1) = f(1) \neq 0$

7. Let $f(x) = \begin{Bmatrix} x^{\alpha} \ln x & , & x > 0 \\ 0 & , & x = 0 \end{Bmatrix}$ Rolle's theorem is applicable to f for $x \in [0,1]$, if $\alpha =$ **[IIT-JEE Screening 2004]**

(a) –2
(b) –1
(c) 0
(d) $\frac{1}{2}$

8. The value of a for which the equation $x^3 - 3x + a = 0$ has two distinct roots in [0, 1] is given by

(a) –1
(b) 1
(c) 3
(d) None of these

9. Let a, be two distinct roots of a polynomial $f(x)$. Then there exists at least one root lying between a and b of the polynomial

(a) $f(x)$
(b) $f'(x)$
(c) $f''(x)$
(d) None of these

10. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + ... + \frac{a_{n-1}}{2} + a_n = 0$. Then the function $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$ has in (0, 1)

(a) At least one zero
(b) At most one zero
(c) Only 3 zeros
(d) Only 2 zeros

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Answer Sheet

Assignment (Basic and Advance Level)

1	2	3	4	5	6	7	8	9	10
b	d	b	a	a	b	d	d	b	a